The Algebra of Expectations

In my paper on random variables I gave examples of random variables created from other random variables. In this paper I prove formulas for computing the expectation and variance of random variables created from other random variables.

Theorem 1

\[ E(X + a) = E(X) + a \], where \( a \) is a constant.

Proof:

\[
E(X + a) = (x_1 + a)p(x_1) + (x_2 + a)p(x_2) + ... + (x_k + a)p(x_k) \\
= x_1p(x_1) + x_2p(x_2) + ... + x_kp(x_k) + a(p(x_1) + p(x_2) + ... + p(x_k)) \\
= E(X) + a
\]

\[ \square \]

Theorem 2

\[ E(aX) = aE(X) \], where \( a \) is a constant

Proof:

\[
E(aX) = ax_1p(x_1) + ax_2p(x_2) + ... + ax_kp(x_k) \\
= a(x_1p(x_1) + x_2p(x_2) + ... + x_kp(x_k)) \\
= aE(X)
\]

\[ \square \]
Theorem 3

\[ E(X_1 + X_2 + \ldots + X_N) = E(X_1) + E(X_2) + \ldots + E(X_N) \]

Proof:

\[
E(X_1 + X_2 + \ldots + X_N) = \sum [X_1(\bullet) + X_2(\bullet) + \ldots + X_N(\bullet)] p(\bullet)
\]

\[
= \sum [X_1(\bullet)p(\bullet) + X_2(\bullet)p(\bullet) + \ldots + X_N(\bullet)p(\bullet)]
\]

\[
= \sum X_1(\bullet)p(\bullet) + \sum X_2(\bullet)p(\bullet) + \ldots + \sum X_N(\bullet)p(\bullet)
\]

\[
= E(X_1) + E(X_2) + \ldots + E(X_N)
\]

Comment

Now you know why I introduced an alternative definition of \( E(X) \) in my paper about random variables. The usual way to prove this would be to prove it for \( X_1 + X_2 \) using a complicated summation and use mathematical induction to prove it for \( X_1 + X_2 + \ldots + X_N \). This way, I can prove it all at once without using mathematical induction.
Theorem 4

If X and Y are independent random variables then

\[ E(XY) = E(X)E(Y) \]

Proof:

let k be the number of values that X can take on and let m be the number of values that Y can take on. Since X and Y are independent we have \( p(x_i, y_j) = p(x_i)p(y_j) \). So running through all the combinations of \( x_i, y_j \)'s we get:

\[
E(XY) = x_1 y_1 p(x_1)p(y_1) + x_1 y_2 p(x_1)p(y_2) + \ldots + x_1 y_m p(x_1)p(y_m) + x_2 y_1 p(x_2)p(y_1) + x_2 y_2 p(x_2)p(y_2) + \ldots + x_2 y_m p(x_2)p(y_m) + x_3 y_1 p(x_3)p(y_1) + x_3 y_2 p(x_3)p(y_2) + \ldots + x_3 y_m p(x_3)p(y_m) + \ldots + x_k y_1 p(x_k)p(y_1) + x_k y_2 p(x_k)p(y_2) + \ldots + x_k y_m p(x_k)p(y_m)
\]

Adding the column totals we get \( E(X)E(Y) \)

So \( E(XY) = E(X)E(Y) \)

\[ \square \]
Theorem 5

\[ V(aX) = a^2 \cdot V(X) \quad \text{Where } a \text{ is a constant} \]

Proof:

\[
V(aX) = \sum_{i=1}^{k} (ax_i - aE(X))^2 p(x_i)
\]

\[
= \sum_{i=1}^{k} a^2(x_i - E(X))^2 p(x_i)
\]

\[
= a^2 \sum_{i=1}^{k} (x_i - E(X))^2 p(x_i)
\]

\[
= a^2 \cdot V(X)
\]

\[ \square \]

Theorem 6

If \( X \) and \( Y \) are independent

\[ E[(X-E(X))(Y-E(Y))] = 0 \]

Proof:

\[
E[(X-E(X))(Y-E(Y))] = E[XY - E(X)Y - E(Y)X + E(X)E(Y)]
\]

\[
= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)
\]

\[
= E(XY) - E(X)E(Y)
\]
So since $X$ and $Y$ are independent, then $E(XY) = E(X)E(Y)$ and $E[(X-E(X))(Y-E(Y))] = 0.$

Theorem 7

$$V(X_1+X_2+ \ldots +X_N) = \sum_{i=1}^{N} V(X_i) + 2 \sum_{i<j} E[(X_i-E(X_i))(X_j-E(X_j))]$$

Proof:

$$V(X_1+X_2+ \ldots +X_N) = E[X_1+X_2+ \ldots +X_N - E(X_1+X_2+ \ldots +X_N)]^2$$

$$= E[(X_1 - E(X_1)) + (X_2 - E(X_2)) + \ldots + (X_N - E(X_N))]^2$$

$$= E[(X_1 - E(X_1))^2 + (X_2 - E(X_2))^2 + \ldots + (X_N - E(X_N))^2$$

$$+ 2 \sum_{i<j} ((X_i - E(X_i))(X_j - E(X_j))]$$

$$= E[X_1 - E(X_1)]^2 + E[X_2 - E(X_2)]^2 + \ldots + E[X_N - E(X_N)]^2$$

$$+ 2 \sum_{i<j} E[(X_i - E(X_i))(X_j - E(X_j))]$$

$$= \sum_{i=1}^{N} V(X_i) + 2 \sum_{i<j} E[(X_i-E(X_i))(X_j-E(X_j))]$$

Theorem 8

If $X_1, X_2, \ldots, X_N$ are pairwise independent, then

$$V(X_1+X_2+ \ldots +X_N) = \sum_{i=1}^{N} V(X_i)$$

Proof:
This follows from Theorem 7 and Theorem 6.

Daniel Daniels          updated 5/20/2020